An Additive Decomposition in Logarithmic Towers and Beyond

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The Additive Decomposition Problem

Let $(\mathscr{F}, ')$ be a differential field.

Examples: $(\mathbb{C}(x), \frac{d}{dx})$, log/exp/algebraic extensions over $\mathbb{C}(x)$.

Let $\mathscr{F}' = \{f' \mid f \in \mathscr{F}\}$ be the integrable space of $\mathscr{F}$.

Problem: Given $f \in \mathscr{F}$, compute $g, r \in \mathscr{F}$ such that

$$f = g' + \underbrace{r}_{\text{remainder}} \equiv r \mod \mathscr{F}'$$

with the following two properties:

- (minimality) r is minimal in some sense,
- (integrability) $f \in \mathscr{F}' \iff r = 0$.

Rational Additive Decomposition

Ostrogradsky (1845) and Hermite (1872): $\mathscr{F} = (\mathbb{C}(x), \frac{d}{dx})$

proper with a squarefree denominator

$\uparrow$

Given $f \in \mathscr{F}$, there exists a simple element $r \in \mathscr{F}$ such that

$$f \equiv \underbrace{r}_{\text{remainder}} \mod \mathscr{F}'.$$

Furthermore, $\int r\, dx$ is elementary over $\mathscr{F}$.

More than 140 years later...

1845,1872	1991	2013	2016	2018	2018	2020
Rational[1]	**Motivation** Creative Telescoping[2]	Hyperexponential[3]	Algebraic[4]	Fuchsian D-finite[5] – D-finite[6]	Straight/Flat Primitive Towers[7]	D-finite[8]

1. Ostrogradsky, Hermite
2. Zeilberger
3. Bostan, Chen, Chyzak, Li, Xin
4. Chen, Kauers, Koutschan
5. Chen, van Hoeij, Kauers, Koutschan
6. Bostan, Chyzak, Lairez, Salvy
7. Chen, Du, Li
8. van der Hoeven

Primitive Towers

$\mathscr{F}$ is a primitive tower if $\mathrm{char}(\mathscr{F}) = 0$ and $\exists\ t_1, \ldots, t_n \in \mathscr{F}$ s.t.

$$\begin{array}{ccccccccc} K_0 & \subset & K_1 & \subset & \cdots & \subset & K_n & = \mathscr{F}, \\ \shortparallel & & \shortparallel & & & & \shortparallel & \\ (C(x), \frac{d}{dx}) & & K_0(t_1) & & & & K_{n-1}(t_n) & \end{array}$$

where $t_i' \in K_{i-1} \setminus K_{i-1}'$ for all $i \in \{1, \ldots, n\}$.

Moreover, $\mathscr{F}$ is logarithmic if $t_i' = \frac{g'}{g}$ for some $g \in K_{i-1}$.

Example:

$$\mathbb{R}(x)\,(\log(x), \log(\log(x)), \mathrm{Li}(x))\,,\ \text{where } \mathrm{Li}(x) = \int \frac{1}{\log(x)}\,dx.$$

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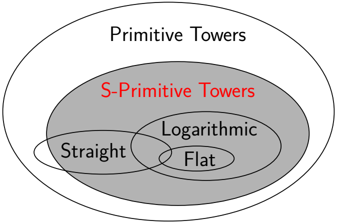
Contribution:



A Direct Sum

Definition. Assume that $\mathscr{F} = K_0(t_1, \ldots, t_n)$. $f \in \mathscr{F}$ is t_i-proper if

$$f \in K_0(t_1, \ldots, t_i) \quad \text{and} \quad \deg_{t_i}(n_f) < \deg_{t_i}(d_f),$$

where n_f and d_f are resp. the numerator and denominator of f.

Proposition.

$$\begin{array}{ccccccccccc} \mathscr{F} = & P_0 & \oplus P_1 \oplus \cdots \oplus P_{n-1} \oplus & P_n & = \bigoplus_{i=0}^{n} P_i \\ & \downarrow & & \downarrow & \\ & K_0[t_1, \ldots, t_n] & & \{f \in K_n \mid f \text{ is } t_n\text{-proper}\} & \end{array}$$

and for all $i \in \{1, 2, \ldots, n-1\}$,

$$P_i = \{p \in K_0(t_1, \ldots, t_i)[t_{i+1}, \ldots, t_n] \mid \text{coeffs}(p) \text{ are } t_i\text{-proper}\}.$$

Matryoshka Decompositions

Definition. Let $\pi_i : \mathscr{F} \to P_i$ be the i-th projection, $\forall\, i \in \{0, \ldots, n\}$. For $f \in \mathscr{F}$, its matryoshka decomposition is

$$f = \sum_{i=0}^{n} \pi_i(f).$$

$$\pi_0(f) + \pi_1(f) + \pi_2(f) + \cdots + \pi_n(f) = f \in \mathscr{F}.$$

An Ordering

Let $\prec$ be the purely lexicographic order such that $t_1 \prec \cdots \prec t_n$.

For $i \in \{0, 1, \ldots n\}$ and $f \in \mathscr{F}$,

$\mathrm{hm}_i(f) :=$ highest monomial in $\pi_i(f)$;

$\mathrm{hm}(f) :=$ highest monomial among $\mathrm{hm}_0(f), \mathrm{hm}_1(f), \ldots, \mathrm{hm}_n(f)$.

Definition. For $f, g \in \mathscr{F}$, we say $f \prec g$ if

- $\deg_{t_n}(d_f) < \deg_{t_n}(d_g)$, or
- $\deg_{t_n}(d_f) = \deg_{t_n}(d_g)$ and $\mathrm{hm}(f) \prec \mathrm{hm}(g)$.

Remainders and S-Primitive Towers

Let $\mathscr{F} = C(x)(t_1, \ldots, t_n)$ be a primitive tower, and $f \in \mathscr{F}$.

Definitions.

- A minimal element in $\{g \in \mathscr{F} \mid g \equiv f \mod \mathscr{F}'\}$ w.r.t. $\prec$ is called a remainder of f.
- For $i \in \{0, 1, \ldots, n\}$ and $t_0 = x$, we say that $f \in \mathscr{F}$ is
  - $\star$ t_i-simple if it is t_i-proper with a squarefree denominator;
  - $\star$ simple if $\pi_i(f)$ is t_i-simple for every i.
- $\mathscr{F}$ is S-primitive if $t_1', \ldots, t_n'$ are all simple.

Example: Log towers are S-primitive.

An Additive Decomposition

Let $\mathscr{F} = C(x)(t_1, \ldots, t_n)$ be an S-primitive tower.

Hermite Reduction [HR]. For any t_i-proper $f \in C(x)(t_1, \ldots, t_i)$, $\exists\ g$ and a t_i-simple h in $C(x)(t_1, \ldots, t_i)$ such that

$$f = g' + h.$$

Integration by Parts [IBP]. Let $h \in C(x)(t_1, \ldots, t_i)$ be simple and

$$M = t_{i+1}^{e_{i+1}} \cdots t_n^{e_n} \quad \text{with} \quad e_{i+1} > 0.$$

- $h \equiv 0 \mod \mathscr{F}' \iff h \in \mathrm{span}_C\{t_1', \ldots, t_n'\}$.
- $h \cdot M \equiv (\text{lower terms}) \mod \mathscr{F}' \iff h \in \mathrm{span}_C\{t_1', \ldots, t_{i+1}'\}$.

Example: HR and IBP

Let $\mathscr{F} = \mathbb{C}(x)(t_1, t_2)$ and $f = -t_2(2t_1x + t_1 - x)/t_1^2 \in \mathscr{F}$, where

$$t_1 = \log(x),\ t_2 = \mathrm{Li}(x).$$

$$\begin{aligned}
f &= \left(\left(-\frac{x^2}{t_1}\right)' - \frac{1}{t_1}\right)t_2 && \text{[HR]}\\
&= \left(-\frac{x^2}{t_1}t_2\right)' + \frac{x^2}{t_1^2} - \frac{1}{t_1}t_2 && \text{[IBP]}\\
&\equiv \frac{x^2}{t_1^2} \mod \mathscr{F}' \\
&\equiv \frac{3x^2}{t_1} \mod \mathscr{F}', && \text{[HR]}
\end{aligned}$$

which gives us a remainder with a lower order than f.

Main Result

Theorem. Let $\mathscr{F} = C(x)(t_1, \ldots, t_n)$ be S-primitive. For $f \in \mathscr{F}$, one can compute $g \in \mathscr{F}$ and a remainder r of f such that

$$f = g' + r.$$

Moreover, $\int f\, dx$ is elementary over $\mathscr{F}$ if and only if

$$r \in \operatorname{span}_C\{t_1', \ldots, t_n'\} + \operatorname{span}_C\{u'/u \mid u \in \mathscr{F}\},$$

provided that C is algebraically closed.

Example 1

$$f = \frac{1}{\log(x)\mathrm{Li}(x)} + \frac{\mathrm{Li}(x) - 2x\log(x)}{(\log(x))^2} + \log(\log(x)).$$

View f as an element of the S-primitive tower

$$\mathscr{F} = \mathbb{C}(x)(\underbrace{\log(x)}_{t_1}, \underbrace{\mathrm{Li}(x)}_{t_2}, \underbrace{\log(\log(x))}_{t_3}),$$

and write $f = 1/(t_1 t_2) + (t_2 - 2xt_1)/t_1^2 + t_3$. By the theorem,

$$f = \Bigg(\underbrace{xt_3 + \frac{t_2^2}{2} - t_2 - \frac{xt_2 + x^2}{t_1}}_{g}\Bigg)' + \underbrace{\frac{1}{t_1 t_2}}_{r}.$$

Since $r \neq 0$, f has no integral in $\mathscr{F}$, but $\int f\,dx = g + \log(t_2)$.

Both Mathematica and Maple leave the integral unevaluated.
Raab's implementation computes the same result.

Example 2

$$f = \frac{\log((x+1)\log(x))}{x\log(x)}$$

$$\mathscr{F} = \mathbb{C}(x)\big(\underbrace{\log(x)}_{t_1},\ \underbrace{\log((x+1)\,t_1)}_{t_2}\big)$$

$$f = \frac{t_2}{xt_1} \in \mathscr{F} \equiv f \mod \mathscr{F}'$$

$$\mathscr{E} = \mathbb{C}(x)\big(\underbrace{\log(x)}_{u_1},\ \underbrace{\log(x+1)}_{u_2},\ \underbrace{\log(u_1)}_{u_3}\big)$$

$$f = \frac{u_2+u_3}{xu_1} \in \mathscr{E} \equiv \frac{u_2}{xu_1} \mod \mathscr{E}'$$

Associated Matrix

Definition. Let $C(x)(t_1, \ldots, t_n)$ be a log tower. The $n \times n$ matrix

$$A = \left(\pi_i(t_j')\right)_{0 \le i \le n-1, 1 \le j \le n}$$

is called the matrix associated to $C(x)(t_1, \ldots, t_n)$.

$$\begin{array}{ccc}
 & & \begin{array}{cccc} t_1' & t_2' & \cdots & t_n' \\ \downarrow & \downarrow & & \downarrow \end{array} \\
\begin{array}{c} P_0 \\ P_1 \\ \vdots \\ P_{n-1} \end{array} & \begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \rightarrow \end{array} & \begin{pmatrix} \star & \star & \cdots & \star \\ & \star & \cdots & \star \\ & & \ddots & \vdots \\ & & & \star \end{pmatrix}
\end{array}$$

Well-Generated Log Towers

Definition. A log tower is well-generated if its associated matrix has the following form,

$$\begin{pmatrix} \bullet & \cdots & \bullet & & & & & & \\ & & & \bullet & \cdots & \bullet & & & \\ & & & & & & \ddots & & \\ & & & & & & \bullet & \cdots & \bullet \end{pmatrix},$$

where the $\bullet$'s form a C-linearly independent list.

Theorem

A logarithmic tower can be differentially embedded into a well-generated one.

Package Demo: AdditiveDecomposition.m

$$f=\frac{\log\left(\frac{x^2+x}{\log(x)}\right)\log\left(\frac{\log(x)}{x}\right)+\left(1-\log(x)+\log\left(\frac{\log(x)}{x}\right)\right)\log\left((2+x)\log(x)\log\left(\frac{\log(x)}{x}\right)\right)}{x\log(x)\log\left(\frac{\log(x)}{x}\right)}$$

$$\mathscr{F}=\mathbb{C}(x)\Big(\underbrace{\log x}_{t_1},\ \underbrace{\log(t_1/x)}_{t_2},\ \underbrace{\log((x^2+x)/t_1)}_{t_3},\ \underbrace{\log((x+2)t_1t_2)}_{t_4}\Big)$$

$$f\equiv\left(\left(\frac{1-t_1}{xt_1t_2}\right)t_4+\text{lower terms}\right)\mod \mathscr{F}'$$

$$\mathscr{E}=\mathbb{C}(x)\Big(\underbrace{\log(x)}_{u_1},\ \underbrace{\log(x+1)}_{u_2},\ \underbrace{\log(x+2)}_{u_3},\ \underbrace{\log(u_1)}_{u_4},\ \underbrace{\log(u_4-u_1)}_{u_5}\Big)$$

$$f\equiv\left(\left(\frac{1}{xu_1}\right)u_3+\text{lower terms}\right)\mod \mathscr{E}'$$

Summary of Results

- We find an additive decomposition in an S-primitive tower and an embedding from a log tower to a well-generated log tower.
- An implementation of our algorithm (as a Mathematica package) with usage examples can be found here:

  https://wongey.github.io/add-decomp-sprimitive/

Future Work:

- Additive decompositions in more general primitive towers and hyperexponential towers
- Existence problem of telescopers in primitive extensions

Thank you!