
Demo of AdditiveDecomposition.m

```
In[1]:= SetDirectory[NotebookDirectory[]];  
<< AdditiveDecomposition.m
```

AdditiveDecomposition.m is written by Hao Du and Elaine Wong, Austrian Academy of Sciences, RICAM, Version 0.2 (July 8, 2020)
→ Type ?AdditiveDecomposition for help

```
In[3]:= ? AdditiveDecomposition
```

Welcome to AdditiveDecomposition.m. This package accompanies the paper **An Additive Decomposition in Logarithmic Towers and Beyond** by Hao Du, Jing Guo, Ziming Li and Elaine Wong.

The main objective of this package is an implementation of the algorithm for decomposing a function in an S -primitive tower into its integrable part and a remainder that is minimal in some sense, both of which are in the same field. If the tower is also logarithmic, then we are able to determine an extension field such that the function can be decomposed further. A function in an S -primitive tower is integrable in the tower if and only if the remainder is equal to zero.

The following commands serve the above objectives: `AddDecompInField` and `WellGenLogTower`.

Common subtasks in the above algorithm are Hermite reduction and computing the matryoshka decomposition. These are also provided in this implementation via `HermiteReduceGeneral` and `ProperDecomposition`, respectively. Please refer to the example notebook for usage examples.

Some other functions that might be useful: `ExtendedEuclidean`, `HeadTerm`, `MonomialIndicator`, `AddDecompLogTower`, `ToGenNames`.

Function to Towers

```
In[4]:= f =
  (Log[(x^2 + x) / Log[x]] Log[Log[x] / x] + (Log[Log[x] / x] + 1 - Log[x]) Log[(x + 2) Log[x] Log[Log[x] / x]]) / (x Log[x] Log[Log[x] / x])
Out[4]= (Log[ $\frac{x + x^2}{\text{Log}[x]}$ ] Log[ $\frac{\text{Log}[x]}{x}$ ] + (1 - Log[x] + Log[ $\frac{\text{Log}[x]}{x}$ ])) Log[(2 + x) Log[x] Log[ $\frac{\text{Log}[x]}{x}$ ]]) / (x Log[x] Log[ $\frac{\text{Log}[x]}{x}$ ]])
```

Identify the logarithmic generators. This is not automated (yet) and must be chosen by the user.

```
In[5]:= genlog = {Log[x], Log[Log[x] / x], Log[(x^2 + x) / Log[x]], Log[(2 + x) Log[x] Log[Log[x] / x]]}
Out[5]= {Log[x], Log[ $\frac{\text{Log}[x]}{x}$ ], Log[ $\frac{x + x^2}{\text{Log}[x]}$ ], Log[(2 + x) Log[x] Log[ $\frac{\text{Log}[x]}{x}$ ]]}
```

We rename the generators so that it is easier to read (automated by ToGenNames). The resulting lists form the input for our main function.

```
In[6]:= {fnew, genlist, derlist} = ToGenNames[f, x, genlog]
Out[6]= { $\frac{t_2 t_3 + (1 - t_1 + t_2) t_4}{t_1 t_2 x}$ , {t1, t2, t3, t4}, { $\frac{1}{x}$ ,  $\frac{1 - t_1}{t_1 x}$ ,  $\frac{-1 + t_1 - x + 2 t_1 x}{t_1 x (1 + x)}$ ,  $\frac{2 - 2 t_1 + 2 t_2 + x - t_1 x + t_2 x + t_1 t_2 x}{t_1 t_2 x (2 + x)}$ }}
```

Now our function no longer contains logs.

```
In[7]:= Print["f="] Print[f];
Print["fnew="] Print[fnew];
```

```
f =
  (Log[ $\frac{x + x^2}{\text{Log}[x]}$ ] Log[ $\frac{\text{Log}[x]}{x}$ ] + (1 - Log[x] + Log[ $\frac{\text{Log}[x]}{x}$ ])) Log[(2 + x) Log[x] Log[ $\frac{\text{Log}[x]}{x}$ ]]) / (x Log[x] Log[ $\frac{\text{Log}[x]}{x}$ ]])
fnew =
   $\frac{t_2 t_3 + (1 - t_1 + t_2) t_4}{t_1 t_2 x}$ 
```

Associated Matrices

We can use the Matryoshka decomposition to construct the associated matrix for the tower (this part is automatically done in the package).

```
In[9]:= AssMat[genlist, derlist]
```

```
Out[9]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{x} & -\frac{1}{x} & \frac{1+2x}{x(1+x)} & \frac{1}{2+x} \\ 0 & \frac{1}{t1x} & -\frac{1}{t1x} & \frac{1}{t1x} \\ 0 & 0 & 0 & \frac{1-t1}{t1t2x} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The elements of the matrix control how we reduce our remainder. This may require a tower extension if there are C-linearly independent elements elsewhere in the matrix. In the well-generated tower case, the extension is optimized for the given function.

```
In[10]:= wglT = WellGenLogTower[x, genlist, derlist];
```

```
%[[4]] // MatrixForm
```

```
Out[11]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{x} & \frac{1}{1+x} & \frac{1}{2+x} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{u1x} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-u1}{u1(-u1+u4)x} \end{pmatrix}$$

We first write our functions in terms of our new generators in the well - generated log tower.

```
In[12]:= fnew  
Thread[genlist → wglT[[1]].wglT[[2]]];  
% // TableForm  
fnew2 = Together[fnew /. %%]
```

Out[12]=
$$\frac{t_2 t_3 + (1 - t_1 + t_2) t_4}{t_1 t_2 x}$$

Out[14]/TableForm=

```
t1 → u1  
t2 → -u1 + u4  
t3 → u1 + u2 - u4  
t4 → u3 + u4 + u5
```

Out[15]=
$$(u_1^2 + u_1 u_2 - u_3 + 2 u_1 u_3 - u_4 - u_2 u_4 - u_3 u_4 - u_5 + 2 u_1 u_5 - u_4 u_5) / (u_1 (u_1 - u_4) x)$$

Main Result: Additive Decomposition

We can find an additive decomposition in the original logarithmic tower and in the corresponding well - generated logarithmic tower (possibly an extension). By the embedding, we get a lower remainder with respect to our ordering.

```
In[16]:= adif = AddDecompInField[fnew, x, genlist, derlist]
adwg = AddDecompInField[fnew2, x, wgl[[1]], wgl[[3]]]
HeadTerm[adif[[2]], genlist]
HeadTerm[adwg[[2]], wgl[[1]]]
```

$$\text{Out[16]= } \left\{ -t_2 + (t_1 + t_2) t_3 + (t_1 + t_2) t_4, \frac{1}{t_1 t_2 x (2 + 3x + x^2)} (-2 t_1 + 2 t_1^2 - 2 t_1^2 t_2 - 2 t_1 t_2^2 + 2 t_4 - 2 t_1 t_4 - 3 t_1 x + 3 t_1^2 x - 6 t_1^2 t_2 x - 6 t_1 t_2^2 x + 3 t_4 x - 3 t_1 t_4 x - t_1 x^2 + t_1^2 x^2 - 3 t_1^2 t_2 x^2 - 3 t_1 t_2^2 x^2 + t_4 x^2 - t_1 t_4 x^2) \right\}$$

$$\text{Out[17]= } \left\{ u_1 + u_4 u_5 + \frac{u_5^2}{2}, \frac{u_1 u_2 - u_3 + 2 u_1 u_3 - u_2 u_4 - u_3 u_4}{u_1 (u_1 - u_4) x} \right\}$$

$$\text{Out[18]= } \left\{ \frac{1 - t_1}{t_1 t_2 x}, t_4 \right\}$$

$$\text{Out[19]= } \left\{ \frac{1}{u_1 x}, u_3 \right\}$$

We compare remainders in the base field. Which is the better remainder?

```
In[20]:= subt = GenSubstitute[x, genlist, derlist];  
subu = GenSubstitute[x, wgt[[1]], wgt[[3]]];  
rem1 = Simplify[Together /@ (adif[[2]] /. subt)]  
rem2 = Simplify[Together /@ (adwg[[2]] /. subu)]
```

```
Out[22]= 
$$\frac{-\text{Log}[x]^2 (2 + 3x + x^2 + (2 + 6x + 3x^2) \text{Log}[\text{Log}[x]]) - (2 + 3x + x^2) (\text{Log}[2 + x] + \text{Log}[\text{Log}[x]] + \text{Log}[\text{Log}[x] - \text{Log}[\text{Log}[x]])] + \text{Log}[x] ((2 + 3x + x^2) \text{Log}[2 + x] + (2 + 3x + x^2) \text{Log}[\text{Log}[x]] + (2 + 6x + 3x^2) \text{Log}[\text{Log}[x]]^2 + (2 + 3x + x^2) (1 + \text{Log}[\text{Log}[x] - \text{Log}[\text{Log}[x]])])}{(x (2 + 3x + x^2) \text{Log}[x] (\text{Log}[x] - \text{Log}[\text{Log}[x])])}$$

```

```
Out[23]= 
$$\frac{(\text{Log}[x] (\text{Log}[1 + x] + 2 \text{Log}[2 + x]) - \text{Log}[1 + x] \text{Log}[\text{Log}[x]] - \text{Log}[2 + x] (1 + \text{Log}[\text{Log}[x]]))}{(x \text{Log}[x] (\text{Log}[x] - \text{Log}[\text{Log}[x])])}$$

```

Compare with Mathematica

We can see that Mathematica leaves the integrable part unevaluated.

In[24]:= `Integrate[Together[f - rem2], x]`

$$\text{Out[24]} = \int \left(\left(-\text{Log}[x] \text{Log}[1+x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] + \text{Log}[2+x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] - 2 \text{Log}[x] \text{Log}[2+x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] + \text{Log}[x] \text{Log}\left[\frac{x+x^2}{\text{Log}[x]}\right] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] + \right. \right. \\ \left. \left. \text{Log}[1+x] \text{Log}[\text{Log}[x]] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] + \text{Log}[2+x] \text{Log}[\text{Log}[x]] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] - \text{Log}\left[\frac{x+x^2}{\text{Log}[x]}\right] \text{Log}[\text{Log}[x]] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] + \text{Log}[x] \right. \right. \\ \left. \left. \text{Log}\left[(2+x) \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right]\right] - \text{Log}[x]^2 \text{Log}\left[(2+x) \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right]\right] - \text{Log}[\text{Log}[x]] \text{Log}\left[(2+x) \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right]\right] + \right. \right. \\ \left. \left. \text{Log}[x] \text{Log}[\text{Log}[x]] \text{Log}\left[(2+x) \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right]\right] + \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] \text{Log}\left[(2+x) \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right]\right] - \right. \right. \\ \left. \left. \text{Log}[\text{Log}[x]] \text{Log}\left[\frac{\text{Log}[x]}{x}\right] \text{Log}\left[(2+x) \text{Log}[x] \text{Log}\left[\frac{\text{Log}[x]}{x}\right]\right] \right) / \left(x \text{Log}[x] (\text{Log}[x] - \text{Log}[\text{Log}[x]]) \text{Log}\left[\frac{\text{Log}[x]}{x}\right] \right) dx$$